CSIDH: 
An Efficient Post-Quantum 
Commutative Group Action 
https://csidh.isogeny.org

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six said
History

1976  Diffie-Hellman: Key exchange using exponentiation in groups (DH)
1985  Koblitz-Miller: Diffie-Hellman style key exchange using multiplication in elliptic curve groups (ECDH)
1990  Brassard-Yung: Generalizes ‘group exponentiation’ to ‘groups acting on sets’ in a crypto context
1994  Shor: Polynomial-time quantum algorithm to break the discrete logarithm problem in any group, quantumly breaking DH and ECDH
1997  Couveignes: Post-quantum isogeny-based Diffie-Hellman-style key exchange using commutative group actions (not published at the time)
2003  Kuperberg: Subexponential-time quantum algorithm to attack cryptosystems based on a hidden shift
History

2004  Stolbunov-Rostovtsev independently rediscover Couveignes’ scheme (CRS)
2006  Charles-Goren-Lauter: Build hash function from supersingular isogeny graph
2010  Childs-Jao-Soukharev: Apply Kuperberg’s (and Regev’s) hidden shift subexponential quantum algorithm to CRS
2011  Jao-De Feo: Build Diffie-Hellman style key exchange from supersingular isogeny graph (SIDH)
2018  De Feo-Kieffer-Smith: Apply new ideas to speed up CRS
2018  Castryck-Lange-Martindale-Panny-Renes: Apply ideas of De Feo, Kieffer, Smith to supersingular curves over $\mathbb{F}_p$ (CSIDH)

(History slides mostly stolen from Wouter Castryck)
Why CSIDH?

- Drop-in post-quantum replacement for (EC)DH
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Why CSIDH?

- Drop-in post-quantum replacement for (EC)DH
- Non-interactive key exchange (full public-key validation); previously an open problem post-quantumly
- Small keys: 64 bytes at conjectured AES-128 security level
- Competitive speed: \( \sim 85 \) ms for a full key exchange
- Flexible:
  - Compatible with 0-RTT protocols such as QUIC
  - [DG] uses CSIDH for ‘SeaSign’ signatures
  - [DGOPS] uses CSIDH for oblivious transfer
  - [FTY] uses CSIDH for authenticated group key exchange
Apart from mathematical background, SIDH and CSIDH actually have very little in common, and are likely to be useful for different applications.

Here is a comparison (mostly stolen from Luca de Feo):

<table>
<thead>
<tr>
<th>Feature</th>
<th>CSIDH</th>
<th>SIDH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Speed (NIST 1)</td>
<td>85ms</td>
<td>≈ 10ms$^1$</td>
</tr>
<tr>
<td>Public key size (NIST 1)</td>
<td>64B</td>
<td>378B</td>
</tr>
<tr>
<td>Key compression (speed)</td>
<td></td>
<td>≈ 15ms</td>
</tr>
<tr>
<td>Key compression (size)</td>
<td>yes (quick and dirty)</td>
<td>222B</td>
</tr>
<tr>
<td>Constant time implementation</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Submitted to NIST</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>Maturity</td>
<td>7 months</td>
<td>7 years</td>
</tr>
<tr>
<td>Best classical attack</td>
<td>$p^{1/4}$</td>
<td>$p^{1/4}$</td>
</tr>
<tr>
<td>Best quantum attack</td>
<td>subexponential</td>
<td>$p^{1/6}$</td>
</tr>
<tr>
<td>Key size scales</td>
<td>quadratically</td>
<td>linearly</td>
</tr>
<tr>
<td>Security assumption</td>
<td>isogeny walk problem</td>
<td>ad hoc</td>
</tr>
<tr>
<td>CPA security</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>CCA security</td>
<td>yes</td>
<td>Fujisaki-Okamoto</td>
</tr>
<tr>
<td>Non-interactive key exchange</td>
<td>yes</td>
<td>unbearably slow</td>
</tr>
<tr>
<td>Signatures (classical)</td>
<td>unbearably slow seconds</td>
<td>seconds</td>
</tr>
<tr>
<td>Signatures (quantum)</td>
<td></td>
<td>still seconds?</td>
</tr>
</tbody>
</table>

$^1$This is a very conservative estimate!
Post-quantum Diffie-Hellman?

Traditionally, Diffie-Hellman works in a group $G$ via the map

$$\mathbb{Z} \times G \rightarrow G$$

$$(x, g) \mapsto g^x.$$
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Shor’s algorithm quantumly computes \( x \) from \( g^x \) in any group in polynomial time.
Post-quantum Diffie-Hellman!

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$$\mathbb{Z} \times G \to G \quad (x, g) \mapsto g^x.$$  

Shor’s algorithm quantumly computes $x$ from $g^x$ in any group in polynomial time.

$\leadsto$ Idea:

Replace exponentiation on the group $G$ by a group action of a group $H$ on a set $S$:

$$H \times S \to S.$$
Square-and-multiply

Suppose $G \cong \mathbb{Z}/23$ and that Alice computes $g^{13}$. 
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Cycles are compatible: [right, then left] = [left, then right], etc.
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Union of cycles: rapid mixing
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CSIDH: Nodes are now elliptic curves and edges are isogenies.
Graphs of elliptic curves

Nodes: Supersingular curves

\[ E \text{, } y^2 = x^3 + Ax^2 + x \text{ over } F_{419}. \]

Edges: 3-, 5-, and 7-isogenies.
Nodes: Supersingular curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$. 
Graphs of elliptic curves

Nodes: Supersingular curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$. Edges: 3-, 5-, and 7-isogenies.
We want to replace the exponentiation map

\[ \mathbb{Z} \times G \rightarrow G \]

\[ (x, g) \mapsto g^x \]

by a group action on a set.

Replace \( G \) by the set \( S \) of supersingular elliptic curves \( E_A : y^2 = x^3 + Ax^2 + x \) over \( \mathbb{F}_{419} \).

Replace \( \mathbb{Z} \) by a commutative group \( H \)… more details to come!

The action of a well-chosen \( h \in H \) on \( S \) moves the elliptic curves one step around one of the cycles.
Graphs of elliptic curves

A 3-isogeny

$E_{51}: y^2 = x^3 + 51x^2 + x \quad \rightarrow \quad E_9: y^2 = x^3 + 9x^2 + x$

$(x, y) \rightarrow \left( \frac{97x^3 - 183x^2 + x}{x^2 - 183x + 97}, \frac{133x^3 + 154x^2 - 5x + 97}{-x^3 + 65x^2 + 128x - 133} \right)$
Diffie-Hellman on ‘nice’ graphs

Alice
$[+,-,+,-]$ 

Bob
$[+,-,+,+]$
Diffie-Hellman on ‘nice’ graphs

Alice

\[ [+,-,+,-] \]

Bob

\[ [+,-,-,+] \]
Diffie-Hellman on ‘nice’ graphs

Alice
\[ [+, -, +, -] \]

Bob
\[ [+, +, -, +] \]
Diffie-Hellman on ‘nice’ graphs

Alice
[+, −, +, −]

Bob
[+, +, −, +]
Diffie-Hellman on ‘nice’ graphs

Alice

$[+, -, +, -]$  

Bob

$[+, +, -, +]$
Diffie-Hellman on ‘nice’ graphs

Alice
[+, −, +, −]

Bob
[+, +, −, +]
Diffie-Hellman on ‘nice’ graphs

\begin{align*}
\text{Alice} & : [+, -, +, -] \\
\text{Bob} & : [+, +, -, +] 
\end{align*}
Diffie-Hellman on ‘nice’ graphs

Alice

Bob

\[ [+, -, +, -] \]

\[ [+, +, -, +] \]
Diffie-Hellman on ‘nice’ graphs

Alice
[+ , − , + , −]

Bob
[+ , + , − , +]
Diffie-Hellman on ‘nice’ graphs

Alice

\[
[+, -, +, -]
\]

Bob

\[
[+, +, -, +]
\]
Diffie-Hellman on ‘nice’ graphs

Alice

\[ [+,-,+,+] \]

Bob

\[ [+,+,-,+] \]
A walkable graph

- Nodes: Supersingular elliptic curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$. 
A walkable graph

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- Edges: 3-, 5-, and 7-isogenies (more details to come).
A walkable graph

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- Edges: 3-, 5-, and 7-isogenies (more details to come).

Important properties for such a walk:

IP1 ➤ The graph is a composition of compatible cycles.
IP2 ➤ We can compute neighbours in given directions.
Towards IP1: Isogeny graphs

First some reminders (see eg. autumn school slides):

- An elliptic curve $E/F_p$ (for $p \geq 5$) is supersingular if $\#E(F_p) = p + 1$. 
Towards IP1: Isogeny graphs

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- An **isogeny** between two elliptic curves $E \to E'$ is a surjective morphism (of abelian varieties) that preserves the identity.
- For elliptic curves $E, E'/\mathbb{F}_p$ and a prime $\ell \neq p$, an **$\ell$-isogeny** $f : E \to E'$ is an isogeny with $\# \ker(f) = \ell$. 
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► If $f : E \to E'$ is an $\ell$-isogeny, there is a unique dual isogeny $f^\vee : E' \to E$ such that $f^\vee \circ f = [\ell]$ is the multiplication-by-$\ell$ map on $E$. 
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► The dual isogeny is also an $\ell$-isogeny.
Towards IP1: Isogeny graphs

Definition
Let $p$ and $\ell$ be distinct primes. The isogeny graph $G_\ell$ containing $E/\mathbb{F}_p$ is the graph with:

- Nodes: elliptic curves $E'/\mathbb{F}_p$ with $\#E(\mathbb{F}_p) = \#E'(\mathbb{F}_p)$ (up to $\mathbb{F}_p$-isomorphism).
- Edges: we draw an edge $E \rightarrow E'$ to represent an $\ell$-isogeny $f : E \rightarrow E'$ together with its dual $\ell$-isogeny.
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- In our example, these are

$G_3$: 

![Graph Diagram]

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Towards IP1: Isogeny graphs

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▶ In our example, these are $G_5$: 

![Diagram of $G_5$]
Towards IP1: Isogeny graphs

Definition

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- Edges: we draw an edge \( E \rightarrow E' \) to represent an \( \ell \)-isogeny \( f : E \rightarrow E' \) together with its dual \( \ell \)-isogeny.

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\[ G_7 : \]
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$$G_3 \cup G_5 \cup G_7$$
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- Generally, the $G_\ell$ look something like

\[ G_3 : \quad \text{and} \quad G_5 : \]
Towards IP1: Endomorphism rings

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- Equivalently: every node in $G_\ell$ should be distance zero from the cycle.
- Two nodes are at different distances from the cycle if and only if they have different endomorphism rings.
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An endomorphism of an elliptic curve $E$ is a morphism $E \to E$ (as abelian varieties).
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Let $E/\mathbb{F}_p$ be an elliptic curve.

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$$P \mapsto nP$$

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  is an endomorphism.

- The Frobenius map

  \[ \pi : \quad E \to E \]
  \[ (x, y) \mapsto (x^p, y^p) \]

  is an endomorphism.
Towards IP1: Endomorphism rings

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The $\mathbb{F}_p$-rational endomorphism ring $\text{End}_{\mathbb{F}_p}(E)$ of an elliptic curve $E/\mathbb{F}_p$ is the set of $\mathbb{F}_p$-rational endomorphisms.
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Example
Let $p > 3$, let $E/\mathbb{F}_p : y^2 = x^3 + Ax^2 + x$ be a supersingular elliptic curve, and let $\pi$ be the Frobenius endomorphism. Then

$$\pi \circ \pi = [-p]$$

and

$$\mathbb{Z}[\sqrt{-p}] \rightarrow \text{End}_{\mathbb{F}_p}(E)$$

$$n \leftrightarrow [n]$$

$$\sqrt{-p} \leftrightarrow \pi$$

extends $\mathbb{Z}$-linearly to a ring homomorphism.
Towards IP1: Group action

For $p \equiv 3 \pmod{8}$ and $p \geq 5$, if $E_A/\mathbb{F}_p : y^2 = x^3 + Ax^2 + x$ is supersingular, then $\text{End}_{\mathbb{F}_p}(E_A) \cong \mathbb{Z}[\sqrt{-p}]$. 
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- Remember: we want to replace exponentiation $\mathbb{Z} \times G \to G$ with a commutative group action $H \times S \to S$. 
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- Remember: we want to replace exponentiation $\mathbb{Z} \times G \to G$ with a commutative group action $H \times S \to S$.
- The set $S$ is the set of supersingular elliptic curves $E_A/\mathbb{F}_p : y^2 = x^3 + Ax^2 + x$ with $p \equiv 3 \pmod{8}$ and $p \geq 5$. 
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- The group \( H = \text{Cl}(\mathbb{Z}[\sqrt{-p}]) \) is the class group of \( \text{End}_{\mathbb{F}_p}(E_A) \) for (every) \( E_A \in S \).
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- Remember: we want to replace exponentiation $\mathbb{Z} \times G \rightarrow G$ with a **commutative group action** $H \times S \rightarrow S$. 
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- The **group** $H = \text{Cl}(\mathbb{Z}[\sqrt{-p}])$ is the class group of $\text{End}_{\mathbb{F}_p}(E_A)$ for (every) $E_A \in S$. 
- What is the action?
Towards IP1: Group action

- Let $I \subset \text{End}_{F_p}(E_A)$ be an ideal.
Towards IP1: Group action

- Let $I \subset \text{End}_{\mathbb{F}_p}(E_A)$ be an ideal.
- Then

$$H_I = \bigcap_{\alpha \in I} \ker(\alpha)$$

is a subgroup of $E(\overline{\mathbb{F}}_p)$. 

Recall that isogenies are uniquely defined by their kernels (cf. First Isomorphism Theorem of Groups).

Define $f_I: E \to E/ H_I$ to be the isogeny from $E$ with kernel $H_I$.

For $[I] \in \text{Cl}(\mathbb{Z}[\sqrt{-p}])$, let $\tilde{I}$ be an integral representative of the ideal class $[I]$. Then

$$\text{Cl}(\mathbb{Z}[\sqrt{-p}]) \times S \to S ([I], E) \mapsto f_{H_{\tilde{I}}}(E)$$

is a free, transitive group action!
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IP1: The graph is a composition of compatible cycles

- The nodes of the graph are the set $S$ of supersingular elliptic curves $E/\mathbb{F}_p : y^2 = x^3 + Ax^2 + x$ with $p \equiv 3 \pmod{8}$ and $p \geq 5$. 
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  is a free, transitive group action.

- Edges are the isogenies $f_{H_i}$ (together with their duals).
The nodes of the graph are the set $S$ of supersingular elliptic curves $E/\mathbb{F}_p : y^2 = x^3 + Ax^2 + x$ with $p \equiv 3 \pmod{8}$ and $p \geq 5$.

The map

$\text{Cl}(\mathbb{Z}[\sqrt{-p}]) \times S \rightarrow S$

$([I], E) \mapsto f_{H_I}(E)$

is a free, transitive group action.

Edges are the isogenies $f_{H_I}$ (together with their duals).

There is a choice of $\ell_1, \ldots, \ell_n$ such that $G_{\ell_1} \cup \cdots \cup G_{\ell_n}$ is a composition of compatible cycles (IP1).
Towards IP2: Choosing a direction

IP2: Compute neighbours in given directions.
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- Our group action was:

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[I] = [\langle \ell, \pi \pm 1 \rangle].
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- Choosing the direction in the graph corresponds to choosing this sign.
Towards IP2: Computing the neighbours

To compute a neighbour of $E$, we have to compute an $\ell$-isogeny from a given elliptic curve. To do this:

- Find a point $P$ of order $\ell$ on $E$. 

Suppose we have found $P = E(F_p)$ of order $(p+1)/2$.

For every odd prime $\ell | (p+1)$, the point $\ell P$ is a point of order $\ell$.
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To compute a neighbour of \( E \), we have to compute an \( \ell \)-isogeny from a given elliptic curve. To do this:

- Find a point \( P \) of order \( \ell \) on \( E \).
- Compute the isogeny with kernel \( \{ P, 2P, \ldots, \ell P \} \) using Vélu’s formulas (implemented in Sage).
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- **Compute the isogeny with kernel $\{P, 2P, \ldots, \ell P\}$ using Vélu’s formulas** (implemented in Sage).
- **Let $E/\mathbb{F}_p$ be supersingular and $p \geq 5$.**
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► Let $E/\mathbb{F}_p$ be supersingular and $p \geq 5$. Then $E(\mathbb{F}_p) \cong C_{p+1}$ or $C_2 \times C_{(p+1)/2}$.
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- For every odd prime $\ell | (p + 1)$, the point $\frac{p+1}{\ell}P$ is a point of order $\ell$.
- Given a $\mathbb{F}_p$-rational point of order $\ell$, the isogeny computations can be done over $\mathbb{F}_p$. 
To compute the neighbours of supersingular $E/F_p$ with $p \geq 5$ in its $\ell$-isogeny graph $G_\ell$ for odd $\ell|(p + 1)$:
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- $1 \in \mathbb{Z}/\ell\mathbb{Z}$ is an eigenvalue of Frobenius on the $\ell$-torsion; the action $[\langle \ell, \pi - 1 \rangle] * E$ gives an $\ell$-isogeny in the ‘+’ direction.
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- If $p \equiv -1 \pmod{\ell}$ then the action $[\langle \ell, \pi + 1 \rangle] \ast E$ gives an $\ell$-isogeny in the ‘−’ direction.
IP2: Computing neighbours in given directions

For which $\ell$ can we (efficiently) compute the neighbours of supersingular $E/\mathbb{F}_p$ in its $\ell$-isogeny graph $G_{\ell}$ for odd $\ell|(p + 1)$?

\[\text{2You still need a little more to get computations for both the + and \textendash\ directions to be over } \mathbb{F}_p\]
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Given the group action as above, Vélu’s formulas give actual isogenies!

With our design choices all isogeny computations are over $\mathbb{F}_p$. ²

²You still need a little more to get computations for both the $+$ and $-$ directions to be over $\mathbb{F}_p$
Representing nodes of the graph

- Every node of $G_{\ell_i}$ is

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$\Rightarrow$ Can compress every node to a single value $A \in \mathbb{F}_p$.
$\Rightarrow$ Tiny keys!
Does any \( A \) work?

\[ \exists p \text{ of all } A \in F_p \text{ are valid keys.} \]

\[ \text{Public-key validation: Check that } E_A \text{ has } p + 1 \text{ points.} \]

\[ \text{Easy Monte-Carlo algorithm: Pick random } P \text{ on } E_A \text{ and check } \left[ p + 1 \right] P = \infty. \]

\[ \text{This algorithm has a small chance of false positives, but we actually use a variant that } \text{proves} \text{ that } E_A \text{ has } p + 1 \text{ points.} \]
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Classical Security

- Security is based on the **isogeny problem**: given two elliptic curves, compute an isogeny between them.
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- Say Alice’s secret is isogeny is of degree $\ell_1^{e_1} \cdots \ell_n^{e_n}$. She knows the path, so can do only small degree isogeny computations, giving complexity $O(\sum e_i \ell_i)$.

An attacker has to compute one isogeny of large degree (cf. isogeny evaluation complexity from David Jao’s talk).

Alternative way of thinking about it: Alice has to compute the isogeny corresponding to one path from $E_0$ to $E_A$, whereas an attacker has compute all the possible paths from $E_0$.

Best classical attacks are (variants of) meet-in-the-middle: Time $O(4\sqrt{p})$. 
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Hidden-shift algorithms: Subexponential complexity (Kuperberg, Regev).
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- Kuperberg’s algorithm [Kup1] requires a **subexponential number of queries**, and a **subexponential number of operations** on a **subexponential number of qubits**.
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- Kuperberg later [Kup2] gave more trade-off options for quantum and classical memory vs. time.
- Childs-Jao-Soukharev [CJS] applied Kuperberg/Regev to CRS – their attack also applies to CSIDH.
- Part of CJS attack computes many paths in superposition.
Quantum Security

- The exact cost of the Kuperberg/Regev/CJS attack is subtle – it depends on:
  - Choice of time/memory trade-off (Regev/Kuperberg)
  - Quantum evaluation of isogenies

(and much more).

\[^4\text{From [BLMP], using query count of [BS]. [BS] also study quantum evaluation of isogenies but their current preprint misses some costs.}\]
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\[765325228976 \approx 0.7 \cdot 2^{40}\] nonlinear bit operations.

For fastest variant of Kuperberg (uses billions of qubits), total cost of CSIDH-512 attack is about \[2^{81}\] qubit operations.

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## Parameters

<table>
<thead>
<tr>
<th>CSIDH-log $p$</th>
<th>intended NIST level</th>
<th>public key size</th>
<th>private key size</th>
<th>time (full exchange)</th>
<th>cycles (full exchange)</th>
<th>stack memory</th>
<th>classical security</th>
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<tbody>
<tr>
<td>CSIDH-512</td>
<td>1</td>
<td>64 b</td>
<td>32 b</td>
<td>85 ms</td>
<td>212e6</td>
<td>4368 b</td>
<td>128</td>
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<tr>
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<td>64 b</td>
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<td>256</td>
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<tr>
<td>CSIDH-1792</td>
<td>5</td>
<td>224 b</td>
<td>112 b</td>
<td></td>
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<td>4368 b</td>
<td>448</td>
</tr>
</tbody>
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Work in progress & future work

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- More applications.
- [Your paper here!]
Thank you!
References

Mentioned in this talk:

**BLMP** Bernstein, Lange, Martindale, and Panny:
*Quantum circuits for the CSIDH: optimizing quantum evaluation of isogenies*
https://quantum.isogeny.org

**BS** Bonnetain, Schrottenloher:
*Quantum Security Analysis of CSIDH and Ordinary Isogeny-based Schemes*
https://ia.cr/2018/537

**CLMPR** Castryck, Lange, Martindale, Panny, Renes:
*CSIDH: An Efficient Post-Quantum Commutative Group Action*

**CJS** Childs, Jao, and Soukharev:
*Constructing elliptic curve isogenies in quantum subexponential time*
https://arxiv.org/abs/1012.4019

**DG** De Feo, Galbraith:
*SeaSign: Compact isogeny signatures from class group actions*
https://ia.cr/2018/824

**DKS** De Feo, Kieffer, Smith:
*Towards practical key exchange from ordinary isogeny graphs*
https://ia.cr/2018/485
References

Mentioned in this talk (contd.):

DOPS Delpech de Saint Guilhem, Orsini, Petit, and Smart:
Secure Oblivious Transfer from Semi-Commutative Masking
https://ia.cr/2018/648

FTY Fujioka, Takashima, and Yoneyama:
One-Round Authenticated Group Key Exchange from Isogenies
https://eprint.iacr.org/2018/1033

MR Meyer, Reith:
A faster way to the CSIDH
https://ia.cr/2018/782

Kup1 Kuperberg:
A subexponential-time quantum algorithm for the dihedral hidden subgroup problem

Kup2 Kuperberg:
Another subexponential-time quantum algorithm for the dihedral hidden subgroup problem
https://arxiv.org/abs/1112.3333

Reg Regev:
A subexponential time algorithm for the dihedral hidden subgroup problem with polynomial space
Further reading:

BIJ  Biasse, Iezzi, Jacobson:
   *A note on the security of CSIDH*
   https://arxiv.org/pdf/1806.03656

DPV  Decru, Panny, and Vercauteren:
   *Faster SeaSign signatures through improved rejection sampling*
   https://eprint.iacr.org/2018/1109

JLLR Jao, LeGrow, Leonardi, Ruiz-Lopez:
   *A polynomial quantum space attack on CRS and CSIDH*
   (MathCrypt 2018)

Credits: thanks to Lorenz Panny for many of these slides, including all of the beautiful pictures.